ON THE QUASIHARMONIC SYSTEMS, CLOSE TO SYSTEMS WITH CONSTANT COEFFICIENTS, IN WHICH PURE IMAGINARY ROOTS OF THE FUNDAMENTAL EQUATION HAVE NONSIMPLE ELEMENTARY DIVISORS

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In this paper we consider the quasiharmonic equation (1.1) in which the coefficients are analytically dependent on a small parameter μ and when $\mu = 0$, it converts into a system with constant coefficients. We assume that among the roots of the fundamental equation $|a_{s\beta} - \delta_{s\beta}\lambda| = 0$ there exist pure imaginary roots and zero roots, which are multiples of one another and differ from each other by a quantity of the form $2\pi p i/\omega$ (ω is the period of the coefficients of the quasi-harmonic systems; p is an integer) and such that not all of the elementary divisors corresponding to these roots are simple.

In this article attention is mainly focused on those characteristics which arise in the examination of the quasiharmonic system in connection with the existence of the nonsimple elementary divisor of the matrix $\| a_{s\beta} - \delta_{s\beta}\lambda\|$.

Using the Newton polygonal method it is possible to establish the dependence between the structure of the matrix $\| a_{s\beta} - \delta_{s\beta}\lambda \|$ and the quantities $\mu^{1/\gamma}$ with respect to the integral powers into which characteristic roots (and exponents) are developed.

For the practical computation of the characteristic exponents, using substitution (2.1), algebraic equations are derived from which one determines the characteristic exponents in first approximation in the presence of elementary divisors of arbitrary power of the matrix $\| a_{s\beta} - \delta_{s\beta}\lambda \|$.

The results obtained are applied to the examination of the stability of periodic solutions of quasilinear systems with many degrees of freedoms and in those special cases when the fundamental equation of the generating systems has pure imaginary and zero roots, the multiples of which are not equal to the number of groups of solutions corresponding to them.

1. Let us consider a quasiharmonic system

$$\frac{dx_s}{dt} = \sum_{\beta=1}^{n} [a_{s\beta} + \mu f_{s\beta}(t, \mu)] x_\beta \tag{1.1}$$

where the functions $f_{s\beta}(t, \mu)$ are periodic with respect to t with period ω, and analytic with respect to the small parameter μ

$$f_{s\beta}(t, \mu) = f_{s\beta}^{(0)} + \mu f_{s\beta}^{(1)} + \cdots$$

If $\mu = 0$ then the system (1.1) reduces to

$$\frac{dx_s^{(0)}}{dt} = \sum_{\beta=1}^{n} a_{s\beta} x_\beta^{(0)} \qquad (a_{s\beta} = \text{const}) \tag{1.2}$$

Let us assume that among the roots of the fundamental equation

$$|a_{s\beta} - \delta_{s\beta}\lambda| = 0 \qquad (\delta_{ss} = 1, \ \delta_{s\beta} = 0 \text{ for } s \neq \beta) \tag{1.3}$$

of the system (1.2) there exist roots with zero real parts; the roots of such type we will call critical. We assume that among the critical roots one encounters both multiple roots and roots which differ from each other by quantities of the form $2\pi p_u i/\omega$ where p_u is an integer, the number of the groups of solutions of system (1.2) corresponding to the critical roots which are not equal in their multiplicity.

We first consider the case of resonance where among the critical roots λ of the fundamental equation (1.3) there exist the zero and pure imaginary roots

$$\lambda_0 = 0, \qquad \lambda_u = \pm \frac{2\pi}{\omega} p_u i \qquad (u = 1, \ldots, r) \tag{1.4}$$

Let q_0 be the highest power of the elementary divisors of the matrix $\| a_{s\beta} - \delta_{s\beta}\lambda \|$ corresponding to the zero root. Let us denote by $e_{0\gamma}$ the number of elementary divisors λ^γ, where $\gamma = 1, \ldots, q_0$ (if there is no elementary divisor from that series for any value of γ then $e_{0\gamma} = 0$). The multiplicity of the zero root k_0 and the number of the groups solutions m_0 which corresponds to them are equal to

$$k_0 = \sum_{\gamma=1}^{q_0} \gamma e_{0\gamma}, \qquad m_0 = \sum_{\gamma=1}^{q_0} e_{0\gamma} \tag{1.5}$$

Similarly, denote by q_u the highest power of elementary divisors, corresponding to each root of the pair $\lambda_u = \pm 2\pi p_u i/\omega$ and $e_{u\gamma}$ as the number of elementary divisors $(\lambda - \lambda_u)^\gamma$ with $\gamma = 1, \ldots, q_u$.

The multiplicity k_u of the root λ_u and the number of the groups of solutions m_u corresponding to them are equal to

$$k_u = \sum_{\gamma=1}^{q_u} \gamma e_{u\gamma}, \qquad m_u = \sum_{\gamma=1}^{q_u} e_{u\gamma} \tag{1.6}$$

Denote by $x_{s1}(t) \ldots x_{sn}(t)$ the solution of the fundamental system (1.1), defined by the initial conditions

$$x_{jj}(0, \mu) = 1, \qquad x_{sj}(0, \mu) = 0 \qquad (s \neq j)$$

Looking for the solution in the form of series

$$x_{sj}(t) = x_{sj}^{(0)}(t) + \mu x_{sj}^{(1)}(t) + \cdots$$

in which the functions $x_{sj}^{(l)}(t)$ satisfy initial conditions

$$x_{jj}^{(0)}(0) = 1, \quad x_{sj}^{(0)}(0) = 0 \quad (s \neq j)$$

$$x_{sj}^{(1)}(0) = x_{sj}^{(2)}(0) = \ldots = 0 \quad (s \neq j \quad s = j)$$

we obtain

$$x_{sj}^{(l)}(t) = \int_0^t \sum_{\beta=1}^n x_{s\beta}(t-\tau) X_{\beta j}^{(l)}(\tau)\, d\tau \qquad \left(X_{sj}^{(l)} = \sum_{i=1}^l \sum_{\beta=1}^n f_{s\beta}^{(i)} x_{\beta j}^{(l-i)}\right)$$

Then, in the limit of the first approximation, the characteristic equation of the system will be written as

$$D(\rho, \mu) = | x_{s\beta}^{(0)}(\omega) + \mu x_{s\beta}^{(1)}(\omega) - \delta_{s\beta}\rho | = 0 \tag{1.7}$$

Let ρ_0 be the root of equation (1.7) for $\mu = 0$

$$D(\rho, 0) = | x_{s\beta}^{(0)}(\omega) - \delta_{s\beta}\rho_0 | = 0 \tag{1.8}$$

To every root λ_0, λ_u of the fundamental equation (1.3) from the series (1.4), there corresponds a root of equation (1.8) equal to unity; consequently, the multiplicity of the root $\rho_0 = 1$ of the equation (1.8) is equal to $k = k_0 + 2k_1 + \ldots + 2k_r$ and the point $\mu = 0$, $\rho = 1$ is critical.

As it is known, for sufficiently small μ, the equation (1.7) has k roots $\rho(\mu)$ for which $\rho(0) = 1$, and these roots will be expanded into series of integral powers of the quantity $\mu^{1/v}$ where $1 < v < k$.

Let us denote by q the highest power of the elementary divisors, corresponding to the roots (1.4), i.e. the largest of the numbers q_0, q_1, $\ldots$, q_r. We derive conditions for which all k roots $\rho(\mu)$, converting for $\mu = 0$ into unity, will be expanded only in integral power of the quantities μ, $\mu^{1/2}$, $\ldots$, $\mu^{1/2}$. Note that to every elementary divisor $(\lambda - \lambda_u)^\gamma$ of the matrix $\| a_{s\beta} - \delta_{s\beta}\lambda \|$ there corresponds the elementary divisor

$(\rho_0 - 1)^\gamma$ of the matrix $\| x_{s\beta}^{(0)}(\omega) - \delta_{s\beta}\rho_0 \|$ and therefore the number of the elementary divisors $(\rho_0 - 1)^\gamma$ is equal to

$$e_\gamma = e_{0\gamma} + 2e_{1\gamma} + \ldots + 2e_{r\gamma} \qquad (\gamma = 1, \ldots, q) \tag{1.9}$$

The multiplicity k of the root ρ_0 of the characteristic equation (1.8) and the number of groups of solutions corresponding to it are

$$k = \sum_{\gamma=1}^{q} \gamma e_\gamma, \qquad m = \sum_{\gamma=1}^{q} e_\gamma \tag{1.10}$$

Let us introduce in place of ρ the variable σ equal to

$$\sigma = \rho - 1 \tag{1.11}$$

The characteristic equation (1.7) changes into the form

$$D(\sigma, \mu) = | x_{s\beta}^{(0)}(\omega) + \mu x_{s\beta}^{(1)}(\omega) - \delta_{s\beta}(1 + \sigma) | = 0 \tag{1.12}$$

From (1.9) and (1.10) follows, that the matrix

$$\| x_{s\beta}^{(0)}(\omega) - \delta_{s\beta}(1 + \sigma) \| \tag{1.13}$$

through elementary transformations, can be reduced to the diagonal matrix

$$\{ C_1 \ldots C_i \ldots C_n \} \tag{1.14}$$

in which the diagonal members C_i are equal to

$$\begin{array}{ll}
C_i = E_i(\sigma) & \text{for } i = 1, \ldots, n-m \\
C_i = \sigma E_i(\sigma) & \text{for } i = n-m+1, \ldots, n-m+e_1 \\
\cdots\cdots & \cdots\cdots \\
C_i = \sigma^\gamma E_i(\sigma) & \text{for } i = n-m+\varkappa_\gamma+1, \ldots, n-m+\varkappa_\gamma+e_\gamma \\
\cdots\cdots & \cdots\cdots \\
C_n = \sigma^q E_n(\sigma) &
\end{array}$$

where $E_i(\sigma)$ are polynomials not reducing to zero for $\sigma = 0$, and the numbers κ are equal to

$$\varkappa_\gamma = e_1 + e_2 + \cdots + e_{\gamma-1} \tag{1.15}$$

Expanding the characteristic determinant (1.12), we represent the function $D(\sigma, \mu)$ in the form of the series

$$D(\sigma, \mu) = \sum_\nu \mu^\nu a_\nu(\sigma) \tag{1.16}$$

where $a_\nu(\sigma)$ are polynomials in σ with constant coefficients:

$$a_\nu(\sigma) = b_\nu \sigma^{\varepsilon_\nu} + b_{\nu 1}\sigma^{\varepsilon_\nu + 1} + \cdots \tag{1.17}$$

and ϵ_ν represents the smallest exponent of the σ in the polynomial $a_\nu(\sigma)$.

We consider each column of the characteristic determinant $D(\sigma, \mu)$ as a sum of two columns. The first column has for its elements the members $x_{s\beta}^{(0)}(\omega) - \delta_{s\beta}(1 + \sigma)$, and the second has members $\mu x_{s\beta}^{(1)}(\omega)$.

Then, any member $\mu^\nu a_\nu(\sigma)$ of the series (1.16) is equal to the sum of the determinants of n-th order, in each of which should be entered ν second columns and $(n - \nu)$ first columns, where the number of the determinants is equal to the number C_n^{ν} consisting of n elements of ν. The sum of the determinants we denote by a symbol

$$\mu^{\nu} a_{\nu}(\sigma) = \sum_{1}^{\chi}(\nu), \qquad \chi = C_n^{\nu} \tag{1.18}$$

where the symbol in the bracket after the summation sign indicates how many second columns will be contained in each of the summed determinants.

In order to examine the nonexplicit functions $D(\sigma, \mu) = 0$ in the neighborhood of the critical point $\mu = 0$, $\sigma = 0$, we apply the Newton polygonal method, from which we determine satisfactorily the smallest exponent of the power ϵ_{ν} for σ in the polynomial $a_{\nu}(\sigma)$. We begin with the polynomial $a_0(\sigma)$ in which only the constant factor differs from the expression $\sigma^k E_1(\sigma)\ldots E_n(\sigma)$. Since $E_i(0) \neq 0$, $i = 1, \ldots, n$, then, among members of the series (1.16) not containing μ, the smallest exponent in the power of σ is equal to k.

Further, we determine the smallest possible exponent ν_1 for μ in the members of the series not containing σ. Evidently, the exponent $\nu_1 > m$. In fact, for $\nu_1 < m$ each determinant of the n-th order would contain the minor of the matrix (1.13) which has an order greater than $(n - m)$. However, as follows from equation (1.14), in the greatest common divisor of such minor will enter σ as one of the factors, with a power equal or greater than unity. We assume that ν_1 takes the smallest possible value $\nu_1 = m$; then for the construction of Newton's polygon, it will be sufficient to determine the lower exponents of the polynomials $a_{\nu}(\sigma)$ for $\nu = 1, \ldots, m - 1$.

For $\nu = m - 1$, in each determinant from the sum (1.18), the minors of the matrix (1.13) of order $n - m + 1$ will enter. The greatest common divisor of those minors, as can be seen from (1.14), is equal to the derivative of the coefficient of σ which does not reduce to zero for $\sigma = 0$. However, the smallest possible power of σ in polynomial $a_{m-1}(\sigma)$ is $\epsilon_{m-1} > 1$. Thus, it is easy to show that for any polynomial $a_{m-i}(\sigma)$ for $i < e_1$ the smallest possible power for σ is $\epsilon_{m-i} > i$.

To pass to the general case, we assume that the exponent $\nu = m - \kappa_{\gamma} - h$ where κ_{γ} is determined according to the expression (1.15), and h and γ take values: $h = 1, \ldots, e_{\gamma}$, $\gamma = 1, \ldots, q$. The orders of the minors of the matrix (1.13) contained in each of the determinants from the sum (1.18) now will be changed from $n - m + \kappa_{\gamma} + 1$ to $n - m + \kappa_{\gamma} + e_{\gamma}$. By increasing the order of the minor to unity, the domain becomes $e_{\gamma} > h > 1$ as follows from equation (1.14); the greatest common divisors of the minor acquire the factor $\sigma^{\gamma} E_i(\sigma)$ where $E_i(0) \neq 0$. However, the last member of the polynomial $a_i(\sigma)$ with the index $i = m - \kappa_\gamma - h$ contains σ to the power

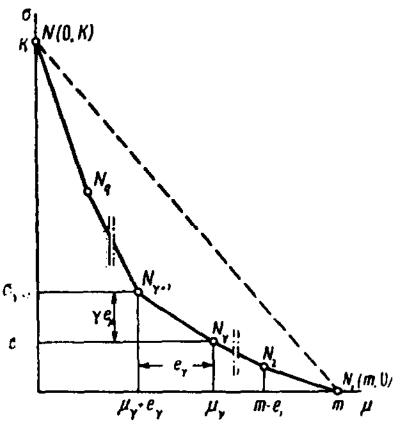
$$\varepsilon_{m-x_\gamma-h} \geqslant \theta(h, \gamma) = e_1 + 2e_2 + \ldots + (\gamma - 1)\, e_{\gamma-1} + \gamma h \qquad (h = 1, \ldots e_\gamma,\ \gamma = 1, \ldots, q)$$

If the last member does not vanish in any of the polynomials $a_\nu(\sigma)$, then the function $D(\sigma, \mu)$ is equal to

$$D(\sigma, \mu) = \sigma^k + b_m\mu^m + \sum_{h,\gamma}\sum b_{m-x_\gamma-h}\mu^{m-x_\gamma-h}\sigma^{\theta(h,\gamma)} + H \quad (1.19)$$

where H is the set of members not belonging to the construction of the Newton polygon.

The Newton polygon for the functions $D(\sigma, \mu)$ is represented in Fig. 1. Through the points $N(0, k)$ and $N_1(m, 0)$, lying on the coordinate axes a dotted line NN_1 is drawn.



$\mu_\gamma = m - x_\gamma$

Fig. 1.

Every member of the series (1.19) represents a point, whose abscissa is equal to the exponent of μ, and whose ordinate is equal to the exponent of σ. Newton's polygon consists of the q segments: N_1N_2, ... $N_\gamma N_{\gamma+1}$ ... N_qN.

As it is known, to each segment of the polygon correspond as many solutions $\sigma(\mu)$, which convert to zero together with μ, as the difference of the ordinates of the end points of the segment. Further, we assume that if the coefficients of the first member of the series corresponding to each segment are all different, then the fractional power of the argument from which the expansion is made for each segment is equal to the tangent $\sigma(\mu)$ of the route angle with respect to the vertical.

From Fig. 1 it follows that the tangent of the angle of the segment $N_\gamma N_{\gamma+1}$ with respect to the vertical plane is equal to $1/\gamma$. Then, the function $\sigma(\mu)$, in the neighborhood of the point $\sigma = \mu = 0$, is expanded in powers of μ (segment N_1N_2), $\mu^{1/2}$ (segment N_2N_3) ... $\mu^{1/q}$ (segment N_qN). The number of expansions corresponding to the segment $N_\gamma N_{\gamma+1}$ is equal to γe_γ and, in general, the number of expansions for the function $\sigma(\mu)$ in the neighborhood of zero is equal to k, as it should be.

From the construction of Newton's polygon for the function $D(\sigma, \mu)$, we conclude that the last member of any polynomial $a_\nu(\sigma)$ does not vanish. However, the Newton polygon for the function $D(\sigma, \mu)$ remains, as shown in Fig. 1, provided the coefficient of the last members, which correspond to the points $N_1, \ldots, N_q$ of the polygon, do not convert to zero, i.e. the following inequality holds for q

$$b_{m-\varkappa_\gamma} \neq 0 \qquad (\gamma = 1, \ldots, q), \tag{1.20}$$

where κ_γ is defined by (1.15). Another form of the conditions (1.20) will be obtained below. Thus, we arrive at the following proposition.

Let us assume that for the quasiharmonic system

$$\frac{dx_s}{dt} = \sum_{\beta=1}^{n} [a_{s\beta} + \mu f_{s\beta}(t, \mu)]\, x_\beta \tag{1.1}$$

the following hold:

(1) The fundamental equation $|a_{s\beta} - \delta_{s\beta}\lambda| = 0$ has among its roots, the zero root, λ_0, and the pure imaginary roots $\lambda_u = \pm 2\pi p_u i/\omega$ where p_u is an integer ($u = 1, \ldots, r$), ω is a period of the function $f_{s\beta}(t,\mu)$, and the sum of the multiplicity of the roots λ_0, λ_u is equal to k.

(2) Elementary divisors of matrix $\| a_{s\beta} - \delta_{s\beta}\lambda\|$ corresponding to the roots λ_0, λ_u are not all simple, and the highest power of these divisors is equal to q.

Then, in order that k characteristic roots $\rho(\mu)$ of the system (1.1), for which $\rho(0) = 1$, be expandable in a series of the quantities μ, $\mu^{1/2}$, ..., $\mu^{1/q}$, the following sufficient conditions must be satisfied:

(a) the first approximation for the group of the roots, $\rho_i(\mu) = 1 + a_i\mu^{1/\gamma} + \ldots$, developed in powers of $\mu^{1/\gamma}$, is different for every γ;

(b) q satisfy inequality (1.20).

In general, the number k of the roots $\rho(\mu)$ developed in series of powers of $\mu^{1/\gamma}$ is equal to γe_γ, where e_γ is the sum of the members of the elementary divisors λ^γ, $(\lambda - \lambda_u)^\gamma$ of the power γ.

If the conditions (a) and (b) are satisfied, we will have the non-degenerate case.

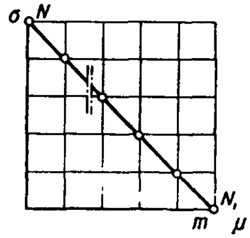


Fig. 2.

Remark 1. From the assumption above, it follows that if the roots λ_0, λ_m have only simple elementary divisors ($\gamma = 1$), and k characteristic roots for which $\rho(0) = 1$ differ little from each other in the first approximation, and the coefficient $b_m \neq 0$, the those characteristic roots are analytic functions in the small parameter μ. This property was indicated by Shumanov. In this case, the polygon of Newton is represented by the line NN_1 (Fig. 2).

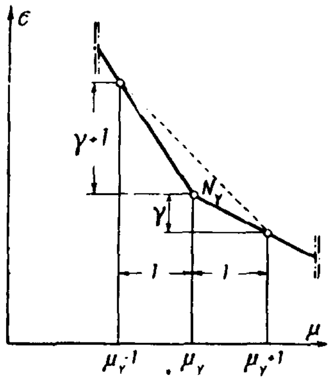


Fig. 3.

Remark 2. If one of the conditions (1.20) is not satisfied, and if for any fixed γ the coefficient b_i, at $i = m - \kappa_\gamma$, is equal to zero, but the coefficients b_{i-1} and b_{i+1} at $m - \kappa_\gamma - 1$ and at $m - \kappa_\gamma + 1$ are different from zero, then in addition to the expansion of the characteristic roots in the nondegenerate case, in powers of μ, $\mu^{1/2}$, ..., $\mu^{1/q}$, $2\gamma + 1$, the roots must be expanded in powers of $\mu^2/(2\gamma + 1)$ (Fig. 3).

So far, we have considered the expansion of the characteristic roots with respect to fractional powers of the small parameter in the case of resonance. If, among the critical roots of the fundamental equation (1.3), there exist pure imaginary roots $\lambda = \pm i\beta$, $\lambda_u = \pm (2\pi p_u/\omega + \beta)i$, $u = 1, \ldots, r$, where β is a real number which is not zero and also is not a multiple of $2\pi/\omega$, then, instead of applying equation (1.1), we use the substitution $\sigma = \rho - e^{i\beta\omega}$; then, it is easy to show that the results obtained are valid in the so-called case of nonresonance.

2. For the practical computation of the characteristic exponent of quasiharmonic system (1.1) Artm'ev has applied the substitution

$$x_s = e^{\alpha t} y_s(t) \tag{2.1}$$

where a is the required characteristic exponent, and $y_s(t)$ is a periodic function of period ω. Here we assume that the fundamental equation (1.3) has neither multiple roots nor roots differing from each other by the quantity $2\pi p_u i/\omega$ (p is an integer).

Shumanov has considered more general cases, assuming that among the roots of the equation (1.3) there can exist roots which are either multiples of each other or which differ from each other by the quantity $\pm 2\pi p_u i/\omega$; the multiplicity of the roots was assumed to be equal to the number of the groups of the solutions of the system (1.2) corresponding to these roots.

We try to remove this last restriction by using the substitution (2.1), and we consider the case when the critical roots of the fundamental equation (1.3) have elementary divisors, which are not all zero. The method of determination of the characteristic exponent developed below is based on the work of Malkin [2].

Furthermore, we assume that to all critical roots (1.4) of the fundamental equation (1.3) for which the sum of the multiplicity is equal to k, there correspond m groups of the solutions; and, in general, the number of elementary divisors λ^γ, $(\lambda - \lambda_u)^\gamma$ $(u = 1, \ldots, r)$ with the power γ is equal to e_γ $(\gamma = 1, \ldots, q)$.

The homogeneous system of differential equations (1.2) has m and only m periodic solutions of period ω, which we denote by ϕ_{si}, where $i = 1, \ldots, m$. We assume that to the periodic solutions $\phi_{s1}, \phi_{s2}, \ldots, \phi_{se}$, for which $e_1 > i > 1$, there corresponds a set of simple elementary divisors λ, $\lambda - \lambda_u$, and that the periodic solutions $\phi_{s,e_1+1}, \ldots, \phi_{s,e_1+e_2}$ correspond to the set of the elementary divisors of the second power λ^2, $(\lambda - \lambda_u)^2$. In general, the periodic solutions, ϕ_{si}, for which the index i takes a value $\kappa_{\gamma+1} > i > \kappa_\gamma$, where κ_γ is defined by (1.15), correspond to the set of the elementary divisors λ^γ, $(\lambda - \lambda_u)^\gamma$ of the power γ. Except for m periodic solutions ϕ_{si}, there correspond to the critical roots λ_0, λ_u, $k - m$ independent particular solutions of the system (1.2) of the following type:

$$\left.\begin{array}{c} \dfrac{t^{\gamma-1}}{(\gamma-1)!}\varphi_{si} + \dfrac{t^{\gamma-2}}{(\gamma-2)!}\varphi_{si}^{(1)} + \ldots + \varphi_{si}^{(\gamma-1)} \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ t\varphi_{si} + \varphi_{si}^{(1)} \end{array}\right\} \begin{array}{l} \text{for } \varkappa_{\gamma+1} \geqslant i > \varkappa_\gamma \\ (\gamma = 2, \ldots, q) \end{array} \tag{2.2}$$

where the functions $\phi_{si}^{(p)}$ are periodic of period ω, satisfying recursive relations

$$\frac{d\varphi_{si}^{(p)}}{dt} = \sum_{\beta=1}^{n} a_{s\beta}\varphi_{\beta i}^{(p)} - \varphi_{si}^{(p-1)}, \quad \varkappa_{\gamma+1} \geqslant i > \varkappa_{\gamma} \qquad (p=1,\ldots,\gamma-1) \tag{2.3}$$

The simplest method for determination of the solutions (2.2) was given by Chetaev [4].

Denote the periodic solutions of the system

$$\frac{dx_s}{dt} + \sum_{\beta=1}^{n} a_{\beta s} x_\beta = 0 \tag{2.4}$$

conjugate to (1.2) through ψ_{si}, where $i = 1, \ldots, m$. As in the previous case, we assume that such periodic solutions ψ_{si}, for which the index i is defined in the range $\kappa_{\gamma+1} > i > \kappa_\gamma$, correspond to the set of the elementary divisors λ^γ, $(\lambda + \lambda_u)^\gamma$ in powers of γ of the matrix $\|-a_{s\beta} - \delta_{s\beta}\lambda\|$. Except for m periodic solution ψ_{si}, there correspond to the critical roots λ_0, λ_u, $k - m$ independent particular solutions of the system (2.4) of the same type as in (2.2), i.e. the solutions

$$\begin{array}{ll} \dfrac{t^{\gamma-1}}{(\gamma-1)!}\psi_{si} + \dfrac{t^{\gamma-2}}{(\gamma-2)!}\psi_{si}^{(1)} + \ldots + \psi_{si}^{(\gamma-1)} & \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots & \text{for } \varkappa_{\gamma+1} \geqslant i > \varkappa_\gamma \\ t\psi_{si} + \psi_{si}^{(1)} & \end{array}$$

where the functions $\psi_{si}^{(p)}$ are periodic with period ω, satisfying the recursive relation

$$\frac{d\psi_{si}^{(p)}}{dt} + \sum_{\beta=1}^{n} a_{\beta s}\psi_{\beta i}^{(p)} + \psi_{si}^{(p-1)} = 0 \qquad (p=1,\ldots,\gamma-1) \tag{2.5}$$

The existence conditions of the periodic solutions of the nonhomogeneous system of equations

$$\frac{dx_s}{dt} = \sum_{\beta=1}^{n} a_{s\beta}x_\beta + f_s(t) \tag{2.6}$$

in the case of resonance, when all the functions $f_s(t)$ are periodic with period ω, as known, can be written [2] in the form:

$$\int_0^\omega \sum_{s=1}^{n} f_s\psi_{sj}\, dt = 0 \tag{2.7}$$

From the relation (2.3) it follows that the functions $\phi_{si}^{(p)}$ are the periodic solutions of the nonhomogeneous system (2.6) for which $f_s(t) = -\phi_{si}^{(p-1)}$. Further, because of relation (2.7), we obtain that in the interval for which $\kappa_{\gamma+1} > i > \kappa_\gamma$ the orthogonality conditions must be satisfied.

$$A_{ji}^{(p-1)} = \int_0^\omega \sum_{s=1}^n \varphi_{si}^{(p-1)}\psi_{sj}\,dt = \omega \sum_{s=1}^n \varphi_{si}^{(p-1)}\psi_{sj} = 0 \quad \begin{pmatrix} p = 1, \ldots, \gamma - 1 \\ j = 1, \ldots, m \end{pmatrix} \tag{2.8}$$

Moreover, for $p = \gamma$ in the interval of i, at least one of the integrals is equal to zero, i.e.

$$A_{ji}^{(\gamma-1)} = \int_0^\omega \sum_{s=1}^n \varphi_{si}^{(\gamma-1)}\psi_{sj}\,dt \neq 0 \quad (j = 1, \ldots, m;\ \gamma = 1, \ldots, q) \tag{2.9}$$

In particular, for any $i < e_1$ one of the sums

$$A_{ji} = \sum_{s=1}^n \varphi_{si}\psi_{sj} \neq 0 \tag{2.10}$$

Similarly, from relations (2.5) and (2.7), we obtain that for any fixed j in the interval $\kappa_{\gamma+1} > j > \kappa_\gamma$,

$$\int_0^\omega \sum_{s=1}^n \psi_{sj}^{(p-1)}\varphi_{si}\,dt = \omega \sum_{s=1}^n \psi_{sj}^{(p-1)}\varphi_{si} = 0 \quad (p = 1, \ldots, \gamma - 1) \tag{2.11}$$

However, at least one of the sums is different from zero:

$$\sum_{s=1}^n \psi_{sj}^{(\gamma-1)}\varphi_{si} \neq 0 \qquad (i = 1, \ldots, m;\ \gamma = 1, \ldots, q)$$

Using the property of the solutions of the conjugate equations (1.2) and (2.4) it is easily shown that

$$\sum_{s=1}^n \varphi_{si}^{(p)}\psi_{sj} = (-1)^p \sum_{s=1}^n \varphi_{si}\psi_{sj}^{(p)}$$

In addition, if we have the form (2.11), we found that for any fixed value j from the interval $\kappa_{\gamma+1} > j > \kappa_\gamma$, the conditions of orthogonality are satisfied, namely,

$$A_{ji}^{(p-1)} = \omega \sum_{s=1}^n \varphi_{si}^{(p-1)}\psi_{sj} = \omega \sum_{s=1}^n \varphi_{si}\psi_{sj}^{(p-1)} = 0 \quad \begin{pmatrix} p = 1, \ldots, \gamma - 1 \\ i = 1, \ldots, m \end{pmatrix} \tag{2.12}$$

From equation (2.8) and equation (2.12) it follows that $A_{ji} = 0$, if at least one of two indices i or j is greater than e_1; $A_{ji}^{(1)} = 0$ if one of the two indices is greater than $e_1 + e_2$; in general $A_{ji}^{(\gamma-2)} = 0$, if $i,\ j < \kappa_\gamma$. In addition, it is evident that $A_{ji}^{(\gamma-2)} \equiv 0$, if $i,\ j < \kappa_{\gamma-1}$, and both of the functions $\phi_{si}^{(\gamma-2)}$ and $\psi_{si}^{(\gamma-2)}$ are identically zero.

Thus, the quantities $A_{ji}^{(\gamma-2)}$ can differ from zero for only those values of the index i, j, which satisfy the inequality $\kappa_\gamma > i,\ j > \kappa_{\gamma-1}$.

Having established the above properties of the quantities $A_{ji}^{(\gamma-2)}$, we pass to the determination of the characteristic exponents of the quasi-harmonic system (1.1). Substituting equation (2.1) into equation (1.1) we have the following system of equations:

$$\frac{dy_s}{dt} = \sum_{\beta=1}^{n} [a_{s\beta} + \mu f_{s\beta}^{(1)}(t) + \mu^2 f_{s\beta}^{(2)}(t) + \ldots] y_\beta - \alpha y_s \tag{2.13}$$

In the nondegenerate cases, as has been shown, the characteristic exponents α will be expanded in the powers of $\mu^{1/\gamma}$, where $\gamma = 1, \ldots, q$, and for any γ the number of the expansions in the $\mu^{1/\gamma}$ is equal to γe_γ. Therefore the characteristic exponent α and the periodic functions y_s can be written in the form of the series

$$\alpha = a_1\mu^{1/\gamma} + a_2\mu^{2/\gamma} + \ldots, \qquad y_s = y_s^{(0)} + \mu^{1/\gamma} y_s^{(1)} + \mu^{2/\gamma} y_s^{(2)} + \ldots \tag{2.14}$$

We begin with the determination of those characteristic exponents, which can be expanded in powers of μ. Substituting the series

$$\alpha = \mu a_1 + \mu^2 a_2 + \ldots, \qquad y_s = y_s^{(0)} + \mu y_s^{(1)} + \mu^2 y_s^{(2)} + \ldots \tag{2.15}$$

into (2.13), we obtain the systems of differential equations

$$\frac{dy_s^{(0)}}{dt} = \sum_{\beta=1}^{n} a_{s\beta} y_\beta^{(0)} \tag{2.16}$$

$$\frac{dy_s^{(1)}}{dt} = \sum_{\beta=1}^{n} a_{s\beta} y_\beta^{(1)} + \sum_{\beta=1}^{n} f_{s\beta}^{(1)} y_\beta^{(0)} - a_1 y_s^{(0)} \tag{2.17}$$

$$\frac{dy_s^{(2)}}{dt} = \sum_{\beta=1}^{n} a_{s\beta} y_\beta^{(2)} + \sum_{\beta=1}^{n} (f_{s\beta}^{(1)} y_\beta^{(1)} + f_{s\beta}^{(2)} y_\beta^{(0)}) - a_1 y_s^{(1)} - a_2 y_s^{(0)} \text{ etc.}$$

The system obtained, (2.16), has a family of periodic solutions

$$y_s^{(0)} = M_1^\circ \varphi_{s1} + \ldots + M_m^\circ \varphi_{sm} \tag{2.19}$$

where M_i° is an arbitrary constant. From the conditions of the periodicity of the functions $y_s^{(1)}$, we obtain

$$P_j = (B_{j1} - a_1 A_{j1}) M_1^\circ + \ldots + (B_{jm} - a_1 A_{jm}) M_m^\circ = 0 \qquad (j = 1, \ldots, m) \tag{2.20}$$

where

$$B_{ji} = \int_0^\omega \sum_s \sum_\beta f_{s\beta}^{(1)} \varphi_{\beta i} \psi_{sj}\, dt, \qquad A_{ji} = \int_0^\omega \sum_{s=1}^{n} \varphi_{si} \psi_{sj}\, dt \tag{2.21}$$

As shown, $A_{ji} = 0$ for $i, j > e_1$. Therefore, the system of equations

(2.20) can be written in the form

$$(B_{j1} - a_1 A_{j1}) M_1^\circ + \ldots + (B_{je_1} - a_1 A_{je_1}) M_{e_1}^\circ + B_{j,e_1+1} M_{e_1+1}^\circ + \ldots +$$
$$+ B_{jm} M_m^\circ = 0 \qquad (j = 1, \ldots, e_1)$$
$$B_{j1} M_1^\circ + \ldots + B_{jm} M_m^\circ = 0 \qquad (j = e_1 + 1, \ldots, m) \tag{2.22}$$

and consequently, the coefficient a_1 is the root of the algebraic equation

$$\Delta_1 = \begin{vmatrix} B_{11} - a_1 A_{11} \ldots B_{1e_1} - a_1 A_{1e_1} & B_{1,e_1+1} \ldots B_{1m} \\ \cdots\cdots\cdots & \cdots\cdots\cdots \\ B_{e_1 1} - a_1 A_{e_1 1} \ldots B_{e_1 e_1} - a_1 A_{e_1 e_1} & B_{e_1, e_1+1} \ldots B_{e_1 m} \\ B_{e_1+1,1} \cdots\cdots & \cdots\cdots B_{e_1+1,m} \\ \cdots\cdots\cdots & \cdots\cdots\cdots \\ B_{m1} \cdots\cdots & \cdots\cdots B_{mm} \end{vmatrix} = 0 \tag{2.23}$$

which is of order equal to e_1. The equations which are obtained will not be satisfied identically if the last $m - e_1$ rows (and columns) of the determinat Δ_1, which do not contain a_1, are linearly independent. Thus, we have obtained e_1 existence conditions for the expansion of the characteristic exponent in powers of μ, in a form different from equation (1.20).

In nondegenerate cases the condition of linear independence of the $m - e_1$ last rows (and columns) of the determinant Δ_1 is satisfied. Moreover, all the e_1 roots of a_1 are simple. Let one of these roots be a_1. Then, among the minors of the $m - 1$ order determinant Δ_1, at least one minor is not equal to zero, and from the system of the equations (2.22) the quantities M_i° can be found, from which one, say M_m°, can be arbitrary. The quantities $M_1^\circ, \ldots, M_{m-1}^\circ$, a_1 are simple solutions of the system (2.22), and for them

$$\frac{\partial (P_1, \ldots, P_m)}{\partial (M_1^\circ, \ldots, M_{m-1}^\circ, a_1)} \neq 0 \tag{2.24}$$

For a nonzero determinant (2.24) we can formally construct the series (2.15), with as many members as we desire.

In fact, the periodic solution for $y_s^{(1)}$ is of the form

$$y_s^{(1)} = M_1^{(1)} \varphi_{s1} + \ldots + M_{m-1}^{(1)} \varphi_{s,m-1} + M_m^\circ \varphi_{sm} + y_s^{(1)*} \tag{2.25}$$

where $M_1^{(1)}, \ldots, M_{m-1}^{(1)}$ are arbitrary constants, and $y_s^{(1)}$ is any particular solution of the periodic system (2.17). Substituting (2.19) and (2.25) into (2.18), we obtain from the conditions of the periodicity of the functions $y_s^{(2)}$

$$\sum_{i=1}^{e_1} (B_{ji} - a_1 A_{ji}) M_i^{(1)} + \sum_{i=e_1+1}^{m-1} B_{ji} M_i^{(1)} - $$
$$- a_2 \sum_{i=1}^{e_1} A_{ji} M_i^{\circ} + C_j = 0 \qquad (j = 1, \ldots, e_1)$$
$$B_{j1} M_1^{(1)} + \ldots + B_{j,m-1} M_{m-1}^{(1)} + C_j = 0 \qquad (j = e_1 + 1, \ldots, m) \tag{2.26}$$

where C_j is an arbitrary constant. From the system of equations (2.26) we can find the quantities $M_1^{(1)}, \ldots, M_{m-1}^{(1)}$ in such a way that the determinant of this system coincides with nonzero determinant (2.24).

Thus, successively one can determine all coefficients a_1, a_2, ... of the series (2.15). Assuming as a_1 all e_1 roots of the equation (2.23) in succession, we obtain e_1 characteristic exponents of the system (1.1) which can be expressed in integer powers of μ.

We pass to the determination of these characteristic exponents which can be expanded in integer powers of $\mu^{1/2}$. The number of such exponents is equal to $2e_2$.

Substituting the series

$$\alpha = \mu^{1/2} a_1 + \mu a_2 + \mu^{3/2} a_3 + , \ldots, \quad y_s = y_s^0 + \mu^{1/2} y_s^{(1)} + \mu y_s^{(2)} + \ldots \tag{2.27}$$

into (2.13) we obtain the system of equations

$$\frac{dy_s^{(0)}}{dt} = \sum_{\beta=1}^{n} a_{s\beta} y_\beta^{(0)}, \qquad \frac{dy_s^{(1)}}{dt} = \sum_{\beta=1}^{n} a_{s\beta} y_\beta^{(1)} - a_1 y_s^{(0)}$$
$$\frac{dy_s^{(2)}}{dt} = \sum_{\beta=1}^{n} a_{s\beta} y_\beta^{(2)} + \sum_{\beta=1}^{n} f_{s\beta}^{(1)} y_\beta^{(0)} - a_1 y_s^{(1)} - a_2 y_s^{(0)} \text{ etc.} \tag{2.28}$$

Assuming for the fundamental solution (2.19), we obtain the conditions of the periodicity for the functions $y_s^{(1)}$:

$$a_1 (M_1^{\circ} A_{j1} + \ldots + M_m^{\circ} A_{jm}) = 0 \qquad (j = 1, \ldots, m) \tag{2.29}$$

Because $A_{ji} = 0$ for $i, j > e_1$, the last $m - e_1$ conditions (2.29) are identically satisfied, and the first e_1 conditions, for $a_1 \neq 0$, take the form

$$M_1^{\circ} A_{j1} + \ldots + M_{e_1} A_{je_1} = 0 \qquad (j = 1, \ldots, e_1) \tag{2.30}$$

Let us show that the determinant $|A_{ji}|$ of the system (2.30) differs from zero. In fact, in the opposite case the equation (2.30) will admit at least one of the solutions $c_1, c_2, \ldots, c_{e1}$ different from the trivial solution $c_1 = c_2 = \ldots = c_{e1} = 0$. Then, assigning to the i a fixed value p_r equal to one of the numbers from the series $1, \ldots, e_1$, we replace the solution of the system (1.2) by the solution $\phi_{sp}{}^* = c_1\phi_{s1} + \ldots + c_{e1}\phi_{se1}$, leaving other solution ϕ_{si} for $i \neq p$ unchanged. For the new system of the fundamental functions $\phi_{s1}, \ldots, \phi_{s,p-1}, \phi_{sp}{}^*, \phi_{s,p+1}, \ldots, \phi_{sm}$ the quantity $A_{jp}{}^*$ is given by

$$A_{jp}{}^* = \int_0^\omega \sum_{s=1}^n \varphi_{sp}{}^*\psi_{sj}\,dt = c_1 A_{j1} + \ldots + c_{e_1}A_{je_1} = 0$$
$$(j = 1, \ldots, e_1)$$

In addition, $A_{jp}{}^* = 0$ for $j > e_1$, then $A_{jp}{}^* = \ldots = A_{mp}{}^* = 0$, which is contrary to the conditions (2.10); this proves that the determinant $|A_{ji}| \neq 0$. Thus, the equation (2.30) can have only the trivial solution $M_1{}^\circ = \ldots = M_{e1}{}^\circ = 0$.

Therefore,

$$y_s{}^\circ = M_{e_1+1}{}^\circ\varphi_{s,e_1+1} + \ldots + M_m{}^\circ\varphi_{sm} \tag{2.31}$$

$$\frac{dy_s{}^{(1)}}{dt} = \sum_{\beta=1}^n a_{s\beta}y_\beta{}^{(1)} - a_1 \sum_{i=e_1+1}^m M_i{}^\circ\varphi_{si} \tag{2.32}$$

where the constants $M_{e_1+1}{}^\circ, \ldots, M_m{}^\circ$ remain undetermined. The family of periodic solutions of the system (2.32) is given by (2.33)

$$y_s{}^{(1)} = M_1{}^{(1)}\varphi_{s1} + \ldots + M_m{}^{(1)}\varphi_{sm} + a_1\,(M_{e_1+1}^\circ\,\varphi_{s,e_1+1}{}^{(1)} + \ldots + M_m{}^\circ\varphi_{sm}{}^{(1)})$$

where $M_1{}^{(1)}, \ldots, M_m{}^{(1)}$ are new arbitrary constants.

Because $A_{ji} = 0$ for $i, j < e_1$ and $A_{ji}{}^{(1)} = 0$ for $i, j < e_1 + e_2$, the conditions of the periodicity of the functions $y_s{}^{(2)}$, with no further transformation, are obtained in the following form:

$$P_j = \sum_{i=e_1+1}^{\varkappa_1} (B_{ji} - a_1{}^2A_{ji}{}^{(1)})\,M_i{}^\circ + \sum_{i=\varkappa_1+1}^{m} B_{ji}M_i{}^\circ = 0 \qquad (j = e_1 + 1, \ldots, e_1 + e_2)$$
$$P_j = B_{je_1+1}M_{e_1+1}{}^\circ + \ldots + B_{jm}\,M_m{}^\circ = 0 \qquad (j = \varkappa_2 + 1, \ldots, m) \tag{2.34}$$

$$P_j = B_{j,e_1+1}M_{e_1+1}{}^\circ + \ldots + B_{jm}M_m{}^\circ - a_1\,(A_{j1}M_1{}^{(1)} + \ldots + A_{je_1}M_{e_1}{}^{(1)})$$
$$(j = 1, \ldots, e_1) \tag{2.35}$$

The constants $M^\circ_{e_1+1}, \ldots, M_m{}^{(\circ)}, M_1{}^{(1)}, \ldots, M_{e_1}{}^{(1)}$ in equations (2.34) and (2.35) are not all zero, only for such values of a_1, for which the the determinant of the system (2.34) becomes zero, i.e. they are roots of the equation

$$\Delta_2 = \begin{vmatrix} B_{e_1+1,e_1+1} - a_1^2 A^{(1)}_{e_1+1,e_1+1} \dots B_{e_1+1,\varkappa_s} - a_1^2 A^{(1)}_{e_1+1,\varkappa_s} B_{e_1+1,\varkappa_s+1} \dots B_{e_1+1,m} \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ B_{\varkappa_s,e_1+1} - a_1^2 A^{(1)}_{\varkappa_s,e_1+1} \dots \quad B_{\varkappa_s\varkappa_s} - a_1^2 A^{(1)}_{\varkappa_s\varkappa_s} B_{\varkappa_s,\varkappa_s+1} \quad \dots B_{\varkappa_s m} \\ B_{\varkappa_s+1,e_1+1} \dots\dots\dots\dots\dots\dots\dots\dots B_{\varkappa_s+1,m} \\ B_{m,e_1+1} \dots\dots\dots\dots\dots\dots\dots\dots B_{mm} \end{vmatrix} = 0 \qquad (2.36)$$

Indeed, if the determinant $\Delta_2 \neq 0$, then from the homogeneous system of equation (2.34) it follows, that $M_{e_1+1}{}^{0} = \dots = M_m{}^{(0)} = 0$; however, if $a_1 \neq 0$, then from the system (2.35) with the determinant $|A_{ji}| \neq 0$, we obtain that $M_1{}^{(1)} = \dots = M_{e_1}{}^{(1)} = 0$.

From equation (2.36) all $2e_2$ characteristic exponents can be determined to the first approximation, if the last $m - \kappa_3$ rows (columns) of the determinant Δ_2 are linearly independent. If, in addition, all roots are simple, then assuming a_1 as one of these roots, we find the quantities $M_{e_1+1}{}^{0}, \dots, M_m{}^{0}$, from which, say $M_m{}^{0}$, can be arbitrarily chosen; whereby

$$\frac{\partial\,(P_{e_1+1} \dots P_m)}{\partial\,(M^{\circ}_{e_1+1} \dots M_{m-1}{}^{\circ}, a_1)} \neq 0 \qquad (2.37)$$

The constants $M_{e_1+1}{}^{0}, \dots, M_m{}^{0}$, obtained from the system of equations (2.35), uniquely determine $M_1(1), \dots, M_{e_1}{}^{(1)}$.

Exactly as in the case for $\gamma = 1$, it is easy to show that for a nonzero determinant of (2.37) the series (2.27) can be formally constructed with an arbitrary number of terms.

Consider the case $\gamma = 3$. Substituting the series

$$\alpha = \mu^{1/3} a_1 + \mu^{2/3} a_2 + \dots, \quad y_s = y_s{}^{(0)} + \mu^{1/3} y_s{}^{(1)} + \mu^{2/3} y_s{}^{(2)} + \dots \qquad (2.38)$$

into (2.13), we obtain

$$\frac{dy_s{}^{(0)}}{dt} = \sum_\beta a_{s\beta} y_\beta{}^{(0)}, \qquad \frac{dy_s{}^{(1)}}{dt} = \sum_\beta a_{s\beta} y_\beta{}^{(1)} - a_1 y_s{}^{(0)}$$

$$\frac{dy_s{}^{(2)}}{dt} = \sum_\beta a_{s\beta} y_\beta{}^{(2)} - a_1 y_s{}^{(1)} - a_2 y_s{}^{(0)}$$

$$\frac{dy_s{}^{(3)}}{dt} = \sum_\beta a_{s\beta} y_\beta{}^{(3)} + \sum_\beta f_{s\beta}{}^{(1)} y_\beta{}^{(0)} - a_1 y_s{}^{(2)} - a_2 y_s{}^{(1)} - a_3 y_s{}^{(0)} \text{ etc.}$$

From the first two systems of equations we find that for the functions $y_s{}^{(0)}$ and $y_s{}^{(1)}$ the equations (2.31) and (2.33) remain valid. The condition of the periodicity of the function $y_s^{(2)}$ is given by

$$a_1 \sum_{i=1}^{m} A_{ji} M_i^{(1)} + a_1^2 \sum_{i=e_1+1}^{m} M_i^{(0)} A_{ji}^{(1)} +$$
$$+ a_2 \sum_{i=e_1+1}^{m} M_i^{\circ} A_{ji} = 0 \qquad (j = 1,\dots, m)$$

The last $m - \kappa_3$ conditions are identically satisfied, but the first κ_3 of these can be written as

$$a_1 (A_{j1} M_1^{(1)} + \dots + A_{je_1} M_{e_1}^{(1)}) = 0, \qquad j = 1,\dots, e_1 \tag{2.39}$$
$$a_1^2 (A_{j,e_1+1}^{(1)} M_{e_1+1}^{\circ} + \dots + A_{j\varkappa_3}^{(1)} M_{\varkappa_3}^{\circ}) = 0, \qquad j = e_1 + 1,\dots, \varkappa_3 \tag{2.40}$$

It is not difficult to show that the determinant of the system of equations (2.40), $|A_{ji}^{(1)}|$, differs from zero. Therefore,

$$M_{e_1+1}^{\circ} = \dots = M_{e_1+e_2}^{\circ} = M_1^{(1)} = \dots = M_{e_1}^{(1)} = 0$$

and the functions $y_s^{(0)}$, $y_s^{(1)}$, $y_s^{(2)}$ will be equal to

$$y_s^{(0)} = \sum_{i=\varkappa_3+1}^{m} M_i^{\circ} \varphi_{si}, \qquad y_s^{(1)} = \sum_{i=e_1+1}^{m} M_i^{(1)} \varphi_{si} + a_1 \sum_{i=\varkappa_3+1}^{m} M_i^{\circ} \varphi_{si}^{(1)}$$

$$y_s^{(2)} = \sum_{i=1}^{m} M_i^{(2)} \varphi_{si} + a_1 \sum_{i=e_1+1}^{m} M_i^{(1)} \varphi_{si}^{(1)} + a_1^2 \sum_{i=\varkappa_3+1}^{m} M_i^{0} \varphi_{si}^{(2)} + a_2 \sum_{\varkappa_3+1}^{m} M_i^{\circ} \varphi_{si}^{(1)}$$

Transforming the conditions of the periodicity of the function we obtain

$$\left.\begin{aligned}
&\sum_{i=\varkappa_3+1}^{m} B_{ji} M_i^{\circ} - a_1 \sum_{i=1}^{e_1} M_i^{(2)} A_{ji} = 0 \quad (j = 1,\dots, e_1)\\
&\sum_{i=\varkappa_3+1}^{m} B_{ji} M_i^{\circ} - a_1^2 \sum_{i=e_1+1}^{\varkappa_3} M_i^{(1)} A_{ji}^{(1)} = 0 \qquad (j = e_1 + 1,\dots, \varkappa_3)
\end{aligned}\right\} \tag{2.41}$$

$$\left.\begin{aligned}
&\sum_{i=\varkappa_3+1}^{m} B_{ji} M_i^{\circ} - a_1^3 \sum_{i=\varkappa_3+1}^{\varkappa_4} M_i^{\circ} A_{ji}^{(2)} = 0 \qquad (j = \varkappa_3 + 1,\dots, \varkappa_4)\\
&\sum_{i=\varkappa_3+1}^{m} B_{ji} M_i^{\circ} = 0 \qquad (j = \varkappa_4 + 1,\dots, m)
\end{aligned}\right\} \tag{2.42}$$

From the system of equations (2.42) it follows, that the constants M_i° differ from zero only for values of a_1, which satisfy the equation

$$\Delta_3 = \begin{vmatrix}
B_{\varkappa_3+1,\varkappa_3+1} - a_1^3 A^{(2)}_{\varkappa_3+1,\varkappa_3+1} & B_{\varkappa_3+1,\varkappa_4} - a_1^3 A^{(2)}_{\varkappa_3+1,\varkappa_4} & B_{\varkappa_3+1,\varkappa_4+1} \dots B_{\varkappa_3+1,m}\\
\dots & \dots & \dots\\
B_{\varkappa_4,\varkappa_3+1} - a_1^3 A^{(2)}_{\varkappa_4,\varkappa_3+1} & B_{\varkappa_4,\varkappa_4} - a_1^3 A^{(2)}_{\varkappa_4,\varkappa_4} & B_{\varkappa_4,\varkappa_4+1} \dots B_{\varkappa_4,m}\\
B_{\varkappa_4+1,\varkappa_3+1} & \dots & \dots B_{\varkappa_4+1,m}\\
B_{m,\varkappa_3+1} & \dots & \dots B_{mm}
\end{vmatrix} = 0 \tag{2.43}$$

where $\kappa_3 = e_1 + e_2$ and $\kappa_4 = e_1 + e_2 + e_3$. If the last $m - \kappa_4$ rows (and columns) of the determinant Δ_3 are linearly independent and all roots of equation (2.43) are simple, then the series (2.38) can be constructed with an arbitrary number of terms. For $\gamma = 3$ we obtain for the first terms a_1 of the series (2.14), the same structure as for $\gamma = 1, 2, 3$. The proof of this statement will not be given.

Until now we have considered the determination of the characteristic exponents of the system (1.1) in the case of resonance. If among the roots of the fundamental equation (1.3) there occur pure imaginary roots of the type $\lambda = \pm\, i\beta$, $\lambda_u = \pm\, (2\pi p_u/\omega + \beta)i$, where β is neither zero nor a multiple of the quantity $2\pi/\omega$, then the characteristic exponents, corresponding to these roots, can be written in the form of the series $a = i\beta + \mu^{1/\gamma}a_1 + \mu^{2/\gamma}a_2 + \ldots$, as described previously, without essential change.

Remark. In the above study we have used the conditions of periodicity (2.7), in order to obtain certain general dependencies. Practically, we can write down the existence conditions for the periodic solutions without recourse to the canonical form or considerations of conjugate systems (which will not be done here).

3. The results obtained will be of use in the investigation of the stability of the periodic solutions of the quasilinear system of the type.

$$\frac{dz_s}{dt} = \sum_{\beta=1}^{n} a_{s\beta} z_\beta + f_s(t) + \mu F_s(t, z) \tag{3.1}$$

where $a_{s\beta}$ are constant coefficients and the functions f_s and F_s are continuous and periodic with respect time t, with period ω. In addition, F_s are analytic in the variables $z_1, \ldots, z_n$. We assume that the corresponding matrix $\| a_{s\beta} - \delta_{s\beta} \|$ has the same structure as an analogous matrix for the quasiharmonic system (1.1), shown above. We restrict ourselves to the case of resonance, i.e. we assume that among the roots of the fundamental equation (1.3) there exist zero roots $\lambda_0 = 0$ and the roots of the form $\lambda_u = \pm\, 2\pi p_u i/\omega$ (p_u is an integer), to which no simple elementary divisors correspond. The real parts of all other roots of equation (1.3) are negative.

Note that the quasilinear systems (3.1), whose pure imaginary roots λ_0, λ_u of the fundamental equation have elementary divisors of the second degree ($q = 2$), occur in many important practical problems, when the frequency of the perturbed forces is considerably greater than the natural frequencies of the system. Such problems arise, in particular, in studying the dynamics of high speed machinery and mechanisms.

Let the system (3.1) have the periodic solution $z_s = \phi_s(t, \mu)$, which is analytic with respect to a small parameter μ, and whose stability is to be studied. Using the variational equations for these solutions, we obtain

$$\frac{dx_s}{dt} = \sum [a_{s\beta} + \mu f_{s\beta}(\mu, t)] x_\beta \qquad \left(f_{s\beta}(\mu, t) = \frac{\partial F_s(t, z, \mu)}{\partial z_\beta} \right) \qquad (3.2)$$

i.e. a quasiharmonic system considered above.

In nondegenerate cases the following propositions are valid.

(1) If the roots λ_0, λ_u, have only simple elementary divisors, then for the asymptotic stability of the periodic solution, it is sufficient that in all roots of equation (2.23) the real part has to be negative. In that case the problem of stability is solved by using the Routh-Hurwitz criterion.

(2) If the critical roots λ_0, λ_u, have elementary divisors of the first and second degree, then for the stability of the periodic solution $z_s = \phi_s(\mu, t)$ it is necessary that the roots of the equation (2.23) have no positive real parts and the square roots of the equation (2.36) are real negatives.

The first condition is obvious; the necessity of the second condition follows from the fact that in equation (2.36) a_1 occurs only in even powers.

Both conditions are necessary, but they are not sufficient for stability of periodic solution for $q = 2$.

(3) If the critical roots λ_0, λ_u, have elementary divisors of the third degree, then the periodic solution $z_s = \phi_s(t, \mu)$ is unstable.

In fact, let $a_1^3 = v$; then for any value of v_1 either complex or real, which satisfies equation (2.43), the coefficient a_1 will take on at least one value, whose real part is positive.

BIBLIOGRAPHY

1. Chbotarev, N.G., Mnogougol'nik N'iutona i ego rol' v sovremennom razvitii matematika (Newton's polygon and its importance in modern mathematical analysis). *Isaak N'iuton. Sbornik statei k trekhsotletiiu so dnia rozhdeniia (Isaac Newton. Collected Works Three Hundredeth Anniversary of his Birth.* Isdatel'stvo Akademii Nauk SSSR, Moscow-Leningrad, 1943.

2. Malkin, I.G., *Nekotorye zadachi teorii nelineinykh kolebanii (Certain Problems in the Theory of Nonlinear Vibrations)*. Gostekhizdat, 1956.

3. Artm'ev, N.A., Metod opredaleniia kharakteristicheskikh pokazatelei i prilozhenie ego k dvum zadacham nebesnoi mekhaniki (Method of determination of the characteristic exponents and its application in celestial mechanics). *Isv. Akad. Nauk SSSR* Vol. 8, No. 2, 1944.

4. Chetaev, N.G., *Ustoichivost' dvizheniia (Stability of Motion)*. (2nd. edition) Gostekhizdat, 1955.

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